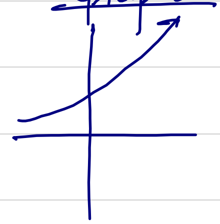
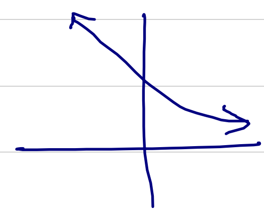
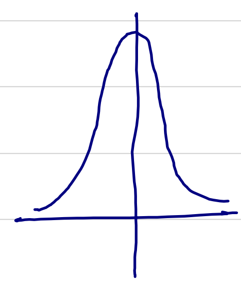
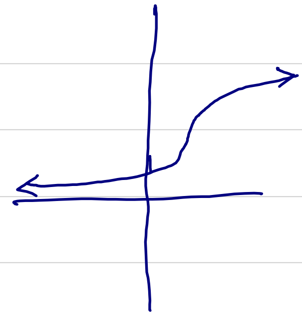
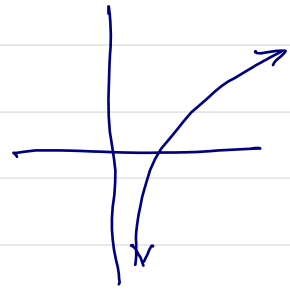
Exponential and Logarithmic Models

Model	Formula	Graph	Application
Exponential growth	$f(t) = ce^{kt}$ $k>0$		Population growth
Exponential decay	$f(t) = ce^{-kt}$ $k>0$		Radioactive decay, carbon dating, depreciation
Gaussian (normal) distribution	$f(t) = ce^{-\frac{(x-a)^2}{k}}$	 Bell Curve	grade distrib, height/weight, IQ tests
Logistic growth	$f(t) = \dfrac{a}{1+ce^{-kt}}$		Predator-Prey model, spread of virus
Logarithmic	$f(t) = a + c\log t$ $f(t) = a + c\ln t$		time to pay off credit card.

I will give you the formulas in word problems. You just need to know how to solve the problems.

<u>Problem</u>. The World population in 2000 was 6.1 billion, and in 2005 it was 6.5 billion. Find the annual growth rate and determine in what year the population will reach 9 billion.

<u>Soln</u>.

We use the exponential growth model.

$$A = Pe^{rt}$$

when
$A = 6.5$ bill.
$P = 6.1$ bill.
$t = 5$ $(2005 - 2000 = 5)$
$r = ?$

$$6.5 \cancel{\text{bill.}} = 6.1 \cancel{\text{bill.}}\, e^{5r}$$

or, $\dfrac{6.5}{6.1} = e^{5r}$

or, $\ln\left(\dfrac{6.5}{6.1}\right) = \ln\left(e^{5r}\right)$

or, $\ln\left(\dfrac{6.5}{6.1}\right) = 5r$

$\therefore r \approx 0.0127$

which is approx. 1.3% per year.

Assuming the growth rate stays the same,

$$A = Pe^{rt}$$

Here, $P = 6.1$ bill.

$r = 1.3\% = 0.013.$

$A = 9$ bill

$t = ?$

So,

$$9 \text{ bill.} = 6.1 \text{ bill. } e^{0.013t}$$

or, $$\frac{9}{6.1} = e^{0.013t}$$

or, $$\ln\left(\frac{9}{6.1}\right) = 0.013t$$

or, $$t = \frac{\ln\left(\frac{9}{6.1}\right)}{0.013}$$

$$\approx 29.9181$$

The world pop. will reach 9 billion in 2030.

Half-Life

The half-life of Uranium-238 is 4.5 billion years. If 98% of uranium-238 remains in a fossil, how old is the fossil?

Soln. The equation for this model is

$$A = Pe^{-rt}$$

First we need to find the rate r.

Let P be some initial value. Then P becomes $\frac{P}{2}$ is 4.5 billion years. Then

$$\frac{P}{2} = P e^{-r(4.5 \text{ bill.})}$$

or, $\frac{1}{2} = e^{-r(4.5 \text{ bill.})}$

or. $\ln\left(\frac{1}{2}\right) = -4.5 \text{ bill.} \cdot r$

or, $r = \frac{\ln\left(\frac{1}{2}\right)}{-4.5 \text{ bill.}}$

$$\approx 0.154033 \ \frac{1}{\text{bill.}}$$

So r is approx. 15.40% but note the unit $\frac{1}{\text{bill.}}$. We dont convert the bill. into zeros because it will be too small.

Since 98% of uranium remains,

$$A = 98\% \text{ of } P$$
$$= \frac{98}{100} \cdot P$$

So.

$$\frac{98\cancel{P}}{100} = \cancel{P} e^{-0.1540t}$$

or, $0.98 = e^{-0.1540t}$

or, $\ln(0.98) = -0.1540\, t$

or, $t = \dfrac{\ln(0.98)}{-0.1540}$

$$\approx 0.131186412$$

Since we don't count $\frac{1}{\text{bill.}}$, this t is in billion years.

$\therefore$ The time require is 131186411 years

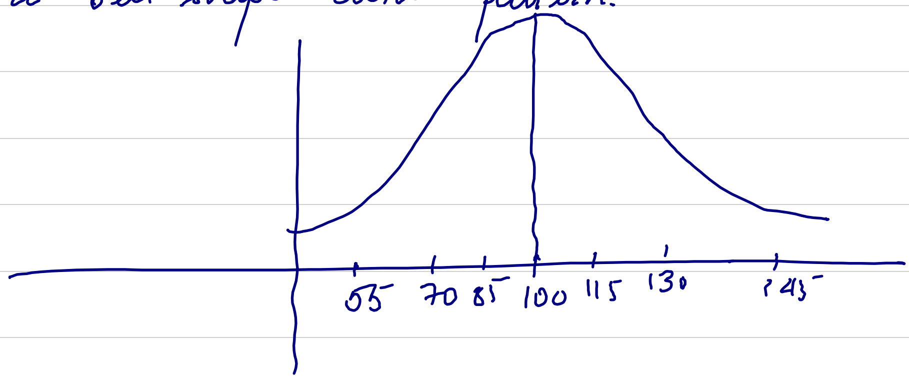
If you want, you can memorize the formula for half life:

$$r = \frac{\ln 2}{h}$$

where h is the time it takes for the material to half.

Gaussian (Normal) Distribution

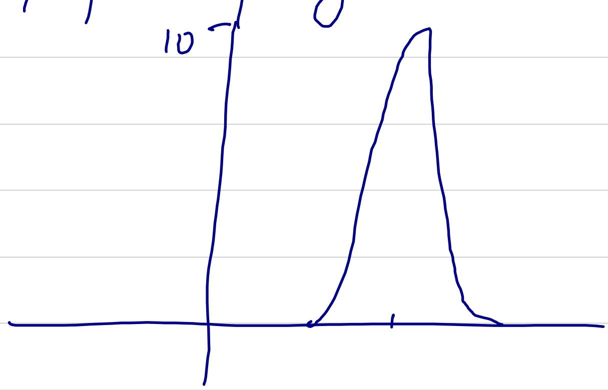
It was first discovered that IQ test scores follow a bell shape curve pattern.



Ex.

Suppose each member of a little league football team is weighed and the weight distribution follows Gaussian model $f(x) = 10e^{-(x-100)^2/25}$

(a) Graph the weight distribution.



(b) What is average weight of team?

Soln: 100 pounds

The book does not explain why? This is taught in a statistics course. For now just note the 100 in the exponent.

(c) Approximately how many boys weigh 95 pounds?

$$f(95) = 10\, e^{-(95-100)^2/25}$$
$$= 10\, e^{-25/25}$$
$$= 10\, e^{-1}$$
$$\approx 3.6788$$

Approx. 4 boys weigh 95 pounds.

__Logistic Growth Models__

In 2008 UCF was largest university in country. The number of undergraduates can be modeled by

$$f(t) = \frac{50{,}000}{1 + 5e^{-0.12t}}$$

where t is time in years and $t = 0$ corresponds to 1970.

(a) How many students attended UCF in 1990? Round to nearest thousand.

__An.__ $t = 20$, $f(20) = \dfrac{50{,}000}{1 + 5e^{-0.12(20)}} \approx 34{,}000$

(b) How many attended in 2000?

__An.__ $t = 30$, $f(30) = \dfrac{50{,}000}{1 + 5e^{-0.12(30)}} \approx 44{,}000$

(c) What is carrying capacity of UCF campus?

__An.__ As t increases, $e^{-0.12t}$ becomes very small. The pop. approaches 50,000.

Logarithmic Models

James owns \$15,000 on his credit card. The annual interest rate is 13% compounded monthly.

(a) Find the time it will take to pay off his credit card if he makes payment of \$200 per month.

Ans - We know

$$t = -\frac{\ln\left(1 - \frac{Pr}{nR}\right)}{n \ln\left(1 + \frac{r}{n}\right)}$$

(this formula will be given to you)

Here R is the periodic payment

r is rate

n is no. of times per year compounded

P is initial amount

So,

$$t = \frac{\ln\left(1 - \frac{15000\ (0.13)}{12\,(200)}\right)}{12 \ln\left(1 + \frac{0.13}{12}\right)}$$

$$\approx 13$$

It will take about 13 years.